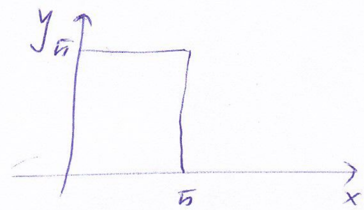
Д/з 1 №1

№1 $\begin{cases} u_t = a^2 \Delta_2 u, & 0<x<\pi, \quad 0<y<\pi, \quad t>0 \\ u|_{t=0} = 3\sin x \sin 5y \\ u|_\gamma = 0 \end{cases}$



$$u(x,y,t) = v(x,y)\cdot T(t)$$

Подставим: $v(x,y)\cdot T'(t) = a^2 T(t)\,\Delta_2 v(x,y)$

$$\Rightarrow \frac{\Delta_2 v}{v} = \frac{T'(t)}{a^2 T(t)} = -\lambda$$

1) $\begin{cases} \Delta_2 v + \lambda v = 0 \\ v|_\gamma = 0 \end{cases}$

2) $T' + \lambda a^2 T = 0$

$\Rightarrow T = C\cdot e^{-\lambda a^2 t}$

$\lambda_{k,m} = k^2 + m^2, \quad k,m \geq 1$

$v_{k,m}(x,y) = \sin kx \cdot \sin my$

$$\Rightarrow u(x,y,t) = \sum_{k,m=1}^{\infty} c_{k,m} \sin kx \cdot \sin my \cdot e^{-\lambda_{k,m} a^2 t}$$

$$u(x,y,0) = \sum_{k,m=1}^{\infty} c_{k,m} \sin kx \cdot \sin my = 3\sin x \sin 5y$$

$\Rightarrow c_{1,5} = 3,$ остальные $c_{k,m} = 0$

$\lambda_{1,5} = 1^2 + 5^2 = 26$

$$\Rightarrow \boxed{u(x,y,t) = 3\sin x \sin 5y \cdot e^{-26a^2 t}}$$

№ 2

$$\begin{cases} u_t = a^2 \Delta_3 u & 0 < x, y z < l, \quad t > 0 \\ u|_{t=0} = u_0 \\ u|_{\partial\Omega} = 0 \end{cases}$$

$$u(x, y, z, t) = V(x, y z) \cdot T(t)$$

Подставляем в ур-ие. $V \cdot T' = a^2 \cdot T \cdot \Delta_3 V \Rightarrow \frac{\Delta_3 V}{V} = \frac{T'}{a^2 T} = -\lambda$

Рассмотрим задачу: $\begin{cases} \Delta_3 V + \quad V = 0 \\ V(x, y, z) \big|_{\partial\Omega} \equiv 0 \end{cases}$

$$V(x, y, z) = X(x) \cdot Y(y) \cdot Z(z)$$

$$X'' Y Z + X Y'' Z + X Y Z'' + \quad X Y Z = 0 \quad \Rightarrow$$

$$\underbrace{\frac{X''}{X}}_{=-\mu} + \underbrace{\frac{Y''}{Y}}_{=-\nu} + \underbrace{\frac{Z''}{Z}}_{=-\omega} + \lambda = 0$$

Для каждого X, Y, Z решаем задачу Штурма-Лиувилля

$$\begin{cases} X''(x) + \mu X(x) = 0 \\ X(0) = 0 \\ X(l) = 0 \end{cases}$$

$$\mu_k = \left(\frac{\pi k}{l}\right)^2$$

$$X_k = \sin\left(\frac{\pi k x}{l}\right)$$

Аналогично решаные: $\nu_m = \left(\frac{\pi m}{l}\right)^2, \quad Y_m = \sin\left(\frac{\pi m y}{l}\right)$

$$\omega_n = \left(\frac{\pi n}{l}\right)^2, \quad Z_n = \sin\left(\frac{\pi n z}{l}\right)$$

$$\Rightarrow V(x, y, z) = \sum_{k,m,n=1}^{\infty} \sin\left(\frac{\pi k x}{l}\right) \cdot \sin\left(\frac{\pi m y}{l}\right) \cdot \sin\left(\frac{\pi n z}{l}\right)$$

$$\lambda_{k,m,n} = \left(\frac{\pi k}{l}\right)^2 + \left(\frac{\pi m}{l}\right)^2 + \left(\frac{\pi n}{l}\right)^2$$

$$T_{k,m,n} = e^{-a^2 \lambda_{k,m,n} t}$$

$$u(x, y, z, t) = \sum_{k,m,n=1}^{\infty} \sin\left(\frac{\pi k x}{l}\right) \cdot \sin\left(\frac{\pi m y}{l}\right) \sin\left(\frac{\pi n z}{l}\right) \cdot c_{k,m,n}\, e^{-\left(\frac{a\pi}{l}\right)^2 (k^2 + m^2 + n^2) t}$$

$$u(x, y, z, 0) = u_0 = \sum_{k,m,n=1}^{\infty} c_{k,m,n} \sin\left(\frac{\pi k x}{l}\right) \sin\left(\frac{\pi m y}{l}\right) \sin\left(\frac{\pi n z}{l}\right)$$

№ 2 (продолжение)

$$C_{kmn} = \frac{1}{\|V_{k,m,n}\|^2} \int_0^l\int_0^l\int_0^l u_0 \cdot \sin\left(\frac{\pi k x}{l}\right)\sin\left(\frac{\pi m y}{l}\right)\cdot\sin\left(\frac{\pi n z}{l}\right) dx\,dy\,dz =$$

$$= \left(\frac{2}{l}\right)^3 u_0 \int_0^l \sin\frac{\pi k x}{l}\,dx \int_0^l \sin\frac{\pi m y}{l}\,dy \int_0^l \sin\frac{\pi n z}{l}\,dz =$$

$$= \left\{ \int_0^l \sin\frac{\pi k x}{l}\,dx = -\frac{\cos\frac{\pi k x}{l}}{\frac{\pi k}{l}}\Big|_0^l = -\frac{1}{\frac{\pi k}{l}} + \frac{\cos \pi k}{\frac{\pi k}{l}} = \begin{cases} -\frac{2l}{\pi k}, & k = 2n+1 \\ 0, & k = 2n, \quad n \in \mathbb{N} \end{cases}\right.$$

$$= \begin{cases} -\frac{64}{\pi^3} \cdot \frac{u_0}{k\cdot m\cdot n}, & \text{если } k, m, n \text{ - нечетные,} \\ 0, & \text{иначе} \end{cases}$$

Ответ: $u(x,y,z,t) = \sum\limits_{k,m,n=1}^{\infty} -\frac{64}{\pi^3}\frac{u_0}{(2k-1)(2m-1)(2n-1)} \cdot$

$\cdot \sin\left(\frac{\pi(2k-1)x}{l}\right)\cdot\sin\left(\frac{\pi(2m-1)y}{l}\right)\cdot\sin\left(\frac{\pi(2n-1)}{l}\right) e^{-\left(\frac{a\pi}{l}\right)^2\left((2k-1)^2+(2m-1)^2+\right.}$

$\left.+(2n-1)^2\right)\cdot t$ =

$$= \boxed{-\frac{64 u_0}{\pi^3}\sum_{k,m,n=1}^{\infty} \frac{\sin\left(\frac{\pi(2k-1)x}{l}\right)}{2k-1}\cdot\frac{\sin\left(\frac{\pi(2m-1)y}{l}\right)}{2m-1}\cdot\frac{\sin\left(\frac{\pi(2n-1)z}{l}\right)}{2n-1}\cdot e^{-\left(\frac{a\pi}{l}\right)^2\left((2k-1)^2+(2m-1)^2+(2n-1)^2\right)t}}$$

№ 3

$$\begin{cases} u_t = a^2 \Delta_3 u(r,t) & 0 \le r < 3 \quad t > 0 \\ u\big|_{t=0} = 2 \\ u\big|_{r=3} = 1 \end{cases}$$

Сделаем замену $u(r,t) = v(r,t) + 1$

$$\begin{cases} v_t = a^2 \Delta_3 v(r,t), & 0 \le r < 3, \quad t > 0 \\ v\big|_{t=0} = 1 \\ v\big|_{r=3} = 0 \end{cases}$$

(эта задача решена на семинаре)

$v(r,t) = w(r) \cdot T(t)$

$w(r) \cdot T'(t) = a^2 \Delta_3 w(r) \cdot T(t)$

$$\frac{\Delta_3 w(r)}{w(r)} = \frac{T'(t)}{a^2 T(t)} = -\lambda$$

$$w_k(r) = \frac{\sin\frac{\pi k r}{R}}{r} = \frac{\sin\frac{\pi k r}{3}}{r} \qquad T_k(t) = e^{-\lambda_k a^2 t}, \quad \lambda_k = \left(\frac{\pi k}{R}\right)^2 = \left(\frac{\pi k}{3}\right)^2$$

$$\Rightarrow v(r,t) = \sum_{k=1}^{\infty} c_k \frac{\sin\frac{\pi k r}{3}}{r} e^{-\left(\frac{\pi k}{3}\right)^2 a^2 t}$$

$$v(r,0) = \sum_{k=1}^{\infty} c_k \frac{\sin\frac{\pi k r}{3}}{r} = 1$$

$$\sum_{k=1}^{\infty} c_k \sin\frac{\pi k r}{3} = r$$

$$c_k = \frac{2}{3}\int_0^3 r \sin\frac{\pi k r}{3}\, dr = \frac{2}{3}\, r \frac{-\cos\left(\frac{\pi k r}{3}\right)}{\frac{\pi k}{3}}\Bigg|_0^3 - \underbrace{\int_0^3 -\frac{\cos\left(\frac{\pi k r}{3}\right)}{\frac{\pi k}{3}}\, dr}_{0, \text{ т.к. } \int \cos x \text{ от } 0 \text{ до } \pi k} =$$

$$= -\frac{6}{\pi k}\cos \pi k = -\frac{6}{\pi k}(-1)^k = \frac{6}{\pi k}(-1)^{k+1}$$

$$\Rightarrow \boxed{u(r,t) = 1 + \sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{6}{\pi k} \frac{\sin\frac{\pi k r}{3}}{r} e^{-\left(\frac{\pi k}{3} a\right)^2 t}}$$

№ 4

Задача Штурма-Лиувилля для оператора Лапласа в прямоуг. парал.

$$\begin{cases} \Delta_3 u(x,y,z) + \lambda u = 0, & 0 \le x \le l_1,\ 0 \le y \le l_2,\ 0 \le z \le l_3 \\ u(x,y,z)\big|_{\partial\Omega} \equiv 0 \end{cases}$$

$$u(x,y,z) = X(x)\cdot Y(y)\cdot Z(z)$$

$$X''Y\cdot Z + XY''Z + XYZ'' + \lambda XYZ = 0 \qquad \Big|: X\cdot Y\cdot Z$$

$$\underbrace{\frac{X''}{X}}_{-\mu} + \underbrace{\frac{Y''}{Y}}_{-\nu} + \underbrace{\frac{Z''}{Z}}_{-\omega} + \lambda = 0$$

Одномерная задача Ш-Л:

$$\begin{cases} X'' + \mu X(x) = 0 \\ X(0) = 0 \\ X(l_1) = 0 \end{cases} \qquad \begin{cases} Y'' + \nu Y = 0 \\ Y(0) = 0 \\ Y(l_2) = 0 \end{cases} \qquad \begin{cases} Z'' + \omega Z = 0 \\ Z(0) = 0 \\ Z(l_3) = 0 \end{cases}$$

$$\mu_k = \left(\frac{\pi k}{l_1}\right)^2 \qquad \nu_k = \left(\frac{\pi k}{l_2}\right)^2 \qquad \omega_k = \left(\frac{\pi k}{l_3}\right)^2$$

$$X_k = \sin\frac{\pi k x}{l_1} \qquad Y_k = \sin\frac{\pi k y}{l_2} \qquad Z_k = \sin\frac{\pi k z}{l_3}$$

$$\lambda_{k,m,n} = \left(\frac{\pi k}{l_1}\right)^2 + \left(\frac{\pi m}{l_2}\right)^2 + \left(\frac{\pi n}{l_3}\right)^2$$

$$u_{k,m,n} = \sin\frac{\pi k x}{l_1}\cdot\sin\frac{\pi m y}{l_2}\cdot\sin\frac{\pi n z}{l_3}$$ - собственные ф-ии

$$\|u_{k,m,n}\|^2 = \int_0^{l_1}\int_0^{l_2}\int_0^{l_3} u_{k,m,n}^2\,dx\,dy\,dz = \int_0^{l_1}\sin^2\frac{\pi k x}{l_1}\,dx\int_0^{l_2}\sin^2\frac{\pi m y}{l_2}\,dy\int_0^{l_3}\sin^2\frac{\pi n z}{l_3}\,dz =$$

$$= \frac{l_1\cdot l_2\cdot l_3}{2^3}$$